EXACT SINGULAR BEHAVIOR OF POSITIVE SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS WITH A HARDY POTENTIAL

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Abstract. In this paper, we study the singular behavior at \( x = 0 \) of positive solutions to the equation

\[-\Delta u = \frac{\lambda}{|x|^2} u - |x|^\sigma u^p, \quad x \in \Omega \setminus \{0\},\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with \( 0 \in \Omega \), and \( p > 1 \), \( \sigma > -2 \) are given constants. For the case \( \lambda \leq (N - 2)^2/4 \), the singular behavior of all the positive solutions is completely classified in the recent paper [5]. Here we determine the exact singular behavior of all the positive solutions for the remaining case \( \lambda > (N - 2)^2/4 \). In sharp contrast to the case \( \lambda \leq (N - 2)^2/4 \), where several converging/blow-up rates of \( u(x) \) are possible as \( |x| \to 0 \), we show that when \( \lambda > (N - 2)^2/4 \), every positive solution \( u(x) \) blows up in the same fashion:

\[
\lim_{|x| \to 0} |x|^{2+\sigma/(p-1)} u(x) = \left[ \lambda + \frac{2 + \sigma}{p-1} \left( \frac{2 + \sigma}{p-1} + 2 - N \right) \right]^{1/(p-1)}.
\]

1. Introduction

In this article, we investigate the singular behavior at \( x = 0 \) of positive solutions to the equation

\[-\Delta u = \frac{\lambda}{|x|^2} u - |x|^\sigma u^p, \quad x \in \Omega \setminus \{0\},\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded smooth domain with \( 0 \in \Omega \), and \( p > 1 \), \( \sigma > -2 \) are given constants. The right-hand side of equation (1.1) contains an inverse square term, which is usually called the Hardy potential.

The study of this kind of equations was motivated by the understanding of certain physical phenomena, such as the interaction among neutral atoms in Thomas-Fermi theory ([1, 2]). The rich phenomena exhibited by the behavior of the solution \( u(x) \) near \( x = 0 \) as the parameters \( p, \sigma \) and \( \lambda \) are varied have made (1.1) an interesting mathematical problem, attracting extensive investigations by many researchers. For example, when \( \lambda = 0 \), the singular behavior of (1.1) (and its variations) is studied in [1, 2, 3, 6, 7, 13, 14]. The case \( \lambda \neq 0 \) but \( \sigma = 0 \) is considered in [4, 10], where the term \( u^p \) is replaced by a more general function \( h(u) \) behaving like \( u^p \); [10] mainly deals the case \( \lambda \leq (N - 2)^2/4 \), while [4] only considers the case \( 0 < \lambda < (N - 2)^2/4 \). A complete classification of the behavior of all the positive solutions of (1.1) at \( x = 0 \) for the case \( \lambda \leq (N - 2)^2/4 \) and \( \sigma > -2 \) is obtained recently in [5], where it even allows \( |x|^\sigma \) to be replaced by a more general function \( b(x) \) and \( u^p \) by a more general \( h(u) \). For the special case (1.1), the results of [5] (see Corollaries 7.4 and 7.5 there) reveal the following rich behavior of the positive solutions of (1.1):

Date: August 19, 2016.
1991 Mathematics Subject Classification. 35J55, 35B40.
Key words and phrases. Hardy potential, isolated singularity, positive solution.

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The research of Y. Du is supported by the Australian Research Council.
Theorem A. Suppose that $-\infty < \lambda < (N - 2)^2/4$ and $u$ is any positive solution of (1.1). Define

$$
\tau := \frac{N - 2}{2} - \sqrt{\frac{(N - 2)^2}{4}} - \lambda, \quad p^* := 1 + \frac{2 + \sigma}{N - 2 - \tau}, \quad p^{**} := 1 + \frac{2 + \sigma}{\tau}.
$$

\[\ell := \lambda + \frac{2 + \sigma}{p - 1} \left( \frac{2 + \sigma}{p - 1} + 2 - N \right).\]

(1) If $1 < p < p^*$, then as $|x| \to 0$, exactly one of the following holds:
\[\alpha. \quad |x|^\tau u(x) \text{ converges to a positive constant};\]
\[\beta. \quad |x|^{N-2-\tau} u(x) \text{ converges to a positive constant};\]
\[\gamma. \quad |x|^{2p-1} u(x) \text{ converges to } \ell^{1-p}.\]

(2) If $(p, \lambda) \in [p^*, +\infty) \times (-\infty, 0) \cup [p^*, p^{**}) \times (0, (N-2)^2/4)$, then (1)\(\alpha\) above holds.

(3) If $p = p^{**}$ and $0 < \lambda < (N-2)^2/4$, then

$$\lim_{|x| \to 0} \left( |x| \left( \frac{1}{|x|} \right)^{(1/(2+\sigma)} \right)^\tau u(x) = \left( \frac{N - 2 - 2\tau}{p - 1} \right)^{(1/(p-1))}.\]

(4) If $p > p^{**}$ and $0 < \lambda < (N-2)^2/4$, then (1)\(\gamma\) above holds.

Theorem B. Suppose that $\lambda = (N - 2)^2/4$ and $u$ is any positive solution of (1.1).

(i) If $1 < p < p^*$, then as $|x| \to 0$, exactly one of the following holds:
\[a. \quad (1)\alpha \text{ in Theorem A holds};\]
\[b. \quad |x|^{N-2} \left[ \log(1/|x|) \right]^{-1} u(x) \text{ converges to a positive constant};\]
\[c. \quad (1)\gamma \text{ in Theorem A holds}.\]

(ii) If $p = p^*$, then

$$\lim_{|x| \to 0} \left| x \right|^{N-2} \left[ \log(1/|x|) \right]^{p-1} u(x) = \left( \frac{2(p + 1)}{(p - 1)^2} \right)^{(1/(p-1))}.\]

(iii) If $p > p^*$, then (1)\(\gamma\) in Theorem A holds.

One naturally asks:

*What is the behavior of an arbitrary positive solution of (1.1) when $\lambda > \frac{(N-2)^2}{4}$?*

The purpose of this paper is to give a complete answer to this question. The answer turns out to be surprisingly simple:

**Theorem 1.1.** Suppose that $\lambda > \frac{(N-2)^2}{4}$ and $u(x)$ is an arbitrary positive solution of (1.1). Then (1)\(\gamma\) in Theorem A holds.

In section 2 below, we will introduce the basic ingredients needed in our proof of Theorem 1.1, which include a comparison principle from [9], approximation of the Hardy constant $\frac{(N-2)^2}{4}$ by suitable first eigenvalues ([15]), a rough estimate for any positive solution $u(x)$ of (1.1) of the form

$$C_1 |x|^{-\frac{2+\sigma}{p-1}} \leq u(x) \leq C_2 |x|^{-\frac{2+\sigma}{p-1}} \text{ near } x = 0,$$

and a uniqueness result for the corresponding boundary value problem of (1.1) with zero Dirichlet boundary conditions on $\partial\Omega$. We also explain the reason for the restriction that $\sigma > -2$ : For $\sigma \leq -2$, we prove that any positive solution of (1.1) stays bounded near $x = 0$. Section 3 is devoted to the proof of Theorem 1.1, which relies on the techniques introduced in section 2 as well as some new ideas.
Exact Singular Behavior of Positive Solutions

Some of our results in section 2 are related to [15], where positive solutions of the Dirichlet problem
\begin{equation}
-\Delta u = \frac{\lambda}{|x|^2} u - a(x) u^p \quad \text{in } \Omega \setminus \{0\}, \quad u|_{\partial \Omega} = 0
\end{equation}
are considered. Here \(a(x)\) is nonnegative and continuous on \(\overline{\Omega}\). If \(a(x)\) is strictly positive in \(\overline{\Omega}\), it is shown in [15] that, for any \(\lambda > (N-2)^2/4\), (1.3) has a minimal positive solution and a maximal positive solution; moreover, any positive solution of (1.3) satisfies (1.2) with \(\sigma = 0\). By the method in the proof of Proposition 2.5 in section 2 below, it can be shown that (1.3) actually has a unique positive solution for any \(\lambda > (N-2)^2/4\). The case \(\Omega_0 := \{x \in \Omega : a(x) = 0\}\) is a smooth connected set inside \(\Omega\) and is also investigated in [15].

Throughout this paper, we always suppose that \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N\) \((N \geq 3)\) and 0 is an interior point of \(\Omega\). For convenience of our statement, we denote
\[
B_{\delta}(0) = \{x \in \mathbb{R}^N : |x| < \delta\}, \quad \Omega^\delta = \Omega \setminus B_{\delta}(0),
\]
\[
B_{\gamma,\tau}(0) = \{x \in \mathbb{R}^N : \tau < |x| < \gamma\}.
\]

2. Preparations

In this section we introduce the basic ingredients to be used in the proof of Theorem 1.1.

2.1. A comparison lemma. The following comparison principle of [9] will be frequently used.

**Lemma 2.1.** Suppose that \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(\alpha(x)\) and \(\beta(x)\) are continuous functions in \(\Omega\) with \(|\alpha|_\infty < \infty\), and \(\beta(x)\) is nonnegative and not identically zero. Let \(u_1, u_2 \in C^1(\Omega)\) be positive in \(\Omega\) and satisfy in the weak sense
\[
\Delta u_1 + \alpha(x) u_1 - \beta(x) g(u_1) \leq 0 \leq \Delta u_2 + \alpha(x) u_2 - \beta(x) g(u_2), \quad x \in \Omega
\]
and
\[
\limsup_{x \to \partial \Omega} (u_2 - u_1) \leq 0,
\]
where \(g(u)\) is continuous and such that \(\frac{g(u)}{u}\) is strictly increasing and nonnegative for \(u\) in the range \(\min\{u_1, u_2\} < u < \max\{u_1, u_2\}\). Then \(u_2 \leq u_1\) in \(\Omega\).

**Remark 2.2.** In Lemma 2.1, we have slightly modified the original statement of Lemma 2.1 in [9], where it does not have “and nonnegative” in \(\frac{g(u)}{u}\) is strictly increasing and nonnegative for \(u\) in the range \(\min\{u_1, u_2\} < u < \max\{u_1, u_2\}\). This addition is needed for the conclusion.

2.2. Approximation of \((N-2)^2/4\) by first eigenvalues. It is well known that \((\frac{N-2}{2})^2\) is the best constant in the Hardy inequality
\[
C \int_\Omega \frac{\phi^2}{|x|^2} dx \leq \int_\Omega |\nabla \phi|^2 dx, \quad \forall \phi \in W^{1,2}_0(\Omega),
\]
in the sense that
\begin{equation}
(N-2)^2 = \inf_{\phi \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 dx}{\int_\Omega \frac{\phi^2}{|x|^2} dx},
\end{equation}
and \((\frac{N-2}{2})^2\) is not attained in \(H^1_0(\Omega)\). The right-side of (2.1) coincides with the variational characterization of the “first eigenvalue” of
\begin{equation}
-\Delta \phi = \frac{\lambda}{|x|^2} \phi \quad \text{in } H^1_0(\Omega).
\end{equation}
Generally, the first eigenvalue is attained by the first eigenfunction though this is not the case here due to the singularity at \(x = 0\).
The next lemma shows that the Hardy constant \((N-2)^2/4\) can be approximated by the first eigenvalue of some suitably modified eigenvalue problems of (2.2), which will be useful later in the paper. Suppose that \(\omega\) is a bounded smooth domain in \(\mathbb{R}^N\). Let \(a(x)\) be a positive continuous function over \(\mathcal{W}\) and \(\lambda_1[a(x), \omega]\) denote the first eigenvalue of

\[-\Delta u = \lambda a(x)u \quad \text{in} \quad H_0^1(\omega).\]

**Lemma 2.3.** For \(\epsilon, \delta > 0\) and \(\Omega^\delta = \Omega \setminus \overline{B}_\delta(0)\), we have

(i) \(\lim_{\epsilon \to 0} \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon}, \Omega \right] = \left( \frac{N-2}{2} \right)^2\),

(ii) \(\lim_{\delta \to 0} \lambda_1 \left[ \frac{1}{|x|^2}, \Omega^\delta \right] = \left( \frac{N-2}{2} \right)^2\).

**Proof.** These results can be found in [15]. For completeness, we give the simple proofs below.

By property of the first eigenvalue, we see that \(\lambda_1 \left[ \frac{1}{|x|^2 + \epsilon}, \Omega \right]\) is increasing in \(\epsilon\). Hence

\[H_0 := \lim_{\epsilon \to 0^+} \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon}, \Omega \right]\]

is well-defined. By virtue of (2.1), for any \(\epsilon > 0\) we have

\[
\lambda_1 \left[ \frac{1}{|x|^2 + \epsilon}, \Omega \right] = \inf_{\phi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla \phi|^2 dx}{\int_{\Omega} \frac{\phi^2}{|x|^2 + \epsilon} dx} \geq \left( \frac{N-2}{2} \right)^2.
\]

Thus, we have

\[\left( \frac{N-2}{2} \right)^2 \leq H_0.\]

Suppose \(\left( \frac{N-2}{2} \right)^2 < H_0\). By the monotonicity property of the first eigenvalue with respect to the weight function, for any \(\epsilon > 0\) we have

\[
\inf_{\phi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla \phi|^2 dx}{\int_{\Omega} \frac{\phi^2}{|x|^2 + \epsilon} dx} = \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon}, \Omega \right] > H_0.
\]

By (2.1) and \(\left( \frac{N-2}{2} \right)^2 < H_0\), we can choose \(\psi \in H_0^1(\Omega)\) such that

\[
\int_{\Omega} |\nabla \psi|^2 dx \leq \int_{\Omega} \frac{\psi^2}{|x|^2 + \epsilon} dx < H_0.
\]

It follows that if we choose \(\epsilon > 0\) sufficiently small, then

\[
\int_{\Omega} |\nabla \psi|^2 dx \leq \int_{\Omega} \frac{\psi^2}{|x|^2 + \epsilon} dx < H_0.
\]

This clearly contradicts (2.4). The proof of (i) is thus complete.

We now prove (ii). By (i), for any \(\gamma > 0\) there exists \(\epsilon_0 > 0\) such that

\[\left( \frac{N-2}{2} \right)^2 \leq \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon_0}, \Omega \right] \leq \left( \frac{N-2}{2} \right)^2 + \gamma/2.
\]

By the monotonicity property and continuous dependence of the first eigenvalue with respect to the domain, there exists \(\delta_0 > 0\) such that for any \(0 < \delta \leq \delta_0\) we have

\[\left( \frac{N-2}{2} \right)^2 \leq \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon_0}, \Omega^\delta \right] \leq \left( \frac{N-2}{2} \right)^2 + \gamma.
\]

So, for any \(0 < \delta \leq \delta_0\) we have

\[
\lambda_1 \left[ \frac{1}{|x|^2}, \Omega^\delta \right] < \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon_0}, \Omega^\delta \right] \leq \left( \frac{N-2}{2} \right)^2 + \gamma.
\]
By (2.3) and the monotone property of the first eigenvalue, for any \( \epsilon, \delta > 0 \) we have
\[
\left( \frac{N - 2}{2} \right)^2 \leq \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon}, \Omega \right] \leq \lambda_1 \left[ \frac{1}{|x|^2 + \epsilon}, \Omega^\delta \right].
\]
Letting \( \epsilon \to 0 \) and using (2.5), we obtain, for \( 0 < \delta \leq \delta_0 \),
\[
\left( \frac{N - 2}{2} \right)^2 \leq \lambda_1 \left[ \frac{1}{|x|^2}, \Omega^\delta \right] \leq \left( \frac{N - 2}{2} \right)^2 + \gamma,
\]
which clearly implies (ii).
\[\square\]

2.3. Rough estimate near \( x = 0 \). In this subsection we establish a rough estimate for positive solutions of (1.1) near the origin.

**Lemma 2.4.** Suppose that \( \lambda > \left( \frac{N - 2}{2} \right)^2 \) and \( \sigma > -2 \), and let \( u(x) \) be an arbitrary positive solution of (1.1). Then there exist positive constants \( C_1, C_2, \delta_0 \) such that
\[
C_2|x|^{-\frac{2 + \sigma}{p - 1}} \leq u(x) \leq C_1|x|^{-\frac{2 + \sigma}{p - 1}}, \quad \forall x \in B_{\delta_0}(0) \setminus \{0\}.
\]

**Proof.** Firstly, we prove that (1.1) has a maximal positive solution \( U \), which satisfies the second inequality in (2.6). For any small \( \delta > 0 \) and positive integer \( n \), it is well known (see, e.g. [8]) that
\[
\left\{ \begin{array}{ll}
- \Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & x \in \Omega^\delta, \\
u = \infty, & x \in \partial B_\delta(0), \\
u = n, & x \in \partial \Omega
\end{array} \right.
\]
(2.7)
has a unique positive solution, and we denote it by \( U^\delta_\sigma \). By Lemma 2.1, \( U^\delta_\sigma \) is increasing in \( \delta \) and increasing in \( n \). Hence \( U_\delta(x) := \lim_{n \to \infty} U^\delta_\sigma(x) \) is nondecreasing in \( \delta \). The existence of \( U_\delta(x) \) is well-known; indeed it is the unique positive solution of (2.7) with \( n \) replaced by \( \infty \) (see [8]). So
\[
U(x) := \lim_{\delta \to 0} U_\delta(x)
\]
is well-defined on \( \Omega \setminus \{0\} \). By standard regularity arguments of elliptic equations, \( U \) is a solution of (1.1) satisfying \( U|_{\partial \Omega} = \infty \), that is
\[
\left\{ \begin{array}{ll}
- \Delta U = \lambda \frac{U}{|x|^2} - |x|^\sigma U^p, & x \in \Omega \setminus \{0\}, \\
U = \infty, & x \in \partial \Omega.
\end{array} \right.
\]
(2.8)
Note that at this moment, we do not know the behavior of \( U(x) \) near \( x = 0 \).

Suppose that \( v \) is an arbitrary positive solution of (1.1). We want to show that \( v(x) \leq U(x) \) in \( \Omega \setminus \{0\} \). Clearly it suffices to show that \( v(x) \leq U_\delta(x) \) in \( \Omega^\delta \) for all small \( \delta > 0 \). To this end, we notice that (see [8]) \( U_\delta(x) = \lim_{\eta \to 0} U^\delta_\eta(x) \), where \( U^\delta_\eta \) is the unique solution of (2.7) with \( n \) replaced by \( \infty \) and \( \Omega \) replaced by \( \Omega_\eta := \{ x \in \Omega : d(x, \partial \Omega) > \eta \} \). Applying Lemma 2.1 we immediately obtain \( v(x) \leq U^\delta_\eta(x) \) in \( \Omega_\eta \setminus \overline{B_\delta}(0) \) for all small positive \( \delta \) and \( \eta \). Letting \( \eta \to 0 \) we obtain \( v(x) \leq U_\delta(x) \) in \( \Omega^\delta \). Letting \( \delta \to 0 \), we have
\[
v(x) \leq U(x), \quad \forall x \in \Omega \setminus \{0\}.
\]
This shows that \( U \) is the maximal positive solution of (1.1).

Next we show that
\[
U(x) \leq C|x|^{-\frac{2 + \sigma}{p - 1}} \quad \text{near} \quad x = 0.
\]
This is achieved by some scaling and comparison arguments. For any \( x_0 \in \Omega \setminus \{0\} \) satisfying \( d(x_0, \partial \Omega) > \frac{|x_0|}{2} \), denote
\[
\mathbb{D}(x_0) = \left\{ x \in \mathbb{R}^N : |x - x_0| < \frac{|x_0|}{2} \right\}.
\]
For any \( x \in \mathbb{D}(x_0) \), we have
\[
(2.9) \quad \frac{3}{2}|x_0| > |x| > \frac{1}{2}|x_0|.
\]
We firstly consider the case \( \sigma \geq 0 \). From (2.9) and the equation of (2.8) it follows that
\[
(2.10) \quad -\Delta U \leq 4\lambda |x_0|^{-2}U - 2^{-\sigma}|x_0|^\sigma U^p, \quad x \in \mathbb{D}(x_0).
\]
Define
\[
V(x) := |x_0|^{\frac{2+\sigma}{p-1}} U \left( x_0 + \frac{|x_0|}{2} x \right), \quad x \in B_1(0).
\]
It is easy to see that \( x_0 + \frac{|x_0|}{2} x \in \mathbb{D}(x_0) \) when \( x \in B_1(0) \). So, by virtue of (2.10) we obtain
\[
(2.11) \quad -\Delta V(x) \leq \lambda V(x) - 2^{-2-\sigma} V(x)^p, \quad x \in B_1(0).
\]
It is well known (see [8]) that
\[
(2.12) \quad \begin{cases} 
-\Delta u = \lambda u - 2^{-2-\sigma} u^p, & x \in B_1(0), \\
u = +\infty, & x \in \partial B_1(0)
\end{cases}
\]
has a unique positive solution, and we denote it by \( V_\infty \). In view of (2.11), we can apply Lemma 2.1 to conclude that \( V(x) \leq V_\infty(x) \) in \( B_1(0) \). So we have
\[
|x_0|^{\frac{2+\sigma}{p-1}} U(x_0) \leq V_\infty(0).
\]
That is
\[
U(x_0) \leq V_\infty(0)|x_0|^{-\frac{2+\sigma}{p-1}}.
\]
By the arbitrariness of \( x_0 \), there exist positive constants \( C_1 \) and \( \delta_0 \) such that
\[
(2.13) \quad U(x) \leq C_1|x|^{-\frac{2+\sigma}{p-1}}, \quad \forall x \in B_{\delta_0}(0) \setminus \{0\}.
\]
When \(-2 < \sigma < 0\), from (2.9) and the equation of (2.8) it follows that
\[
-\Delta U \leq 4\lambda |x_0|^{-2}U - 3^{\sigma}2^{-\sigma}|x_0|^\sigma U^p, \quad x \in \mathbb{D}(x_0).
\]
So,
\[
(2.14) \quad -\Delta V(x) \leq \lambda V(x) - 2^{-2-\sigma} 3^{-\sigma} V(x)^p, \quad x \in B_1(0).
\]
Therefore, we can also obtain that there exist positive constants \( C_1 \) and \( \delta_0 \) such that \( U \) satisfies (2.13). The second inequality in (2.6) clearly follows directly from (2.13).

To prove the first inequality in (2.6), we start by showing that (1.1) has a minimal positive solution. Since \( \lambda > \left(\frac{N-2}{2}\right)^2 \), by Lemma 2.3, there exists \( \tau_0 > 0 \) satisfying \( \lambda_1 \left[ \frac{1}{|x|^2}, \Omega^\delta \right] < \lambda \) for any \( 0 < \delta \leq \tau_0 \). So,
\[
(2.15) \quad \begin{cases} 
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & x \in \Omega^\delta, \\
u = 0, & x \in \partial\Omega^\delta
\end{cases}
\]
has a unique positive solution $u_\delta \in C^2(\Omega^\delta)$. It follows from Lemma 2.1 that $U_{\delta_1} \geq u_{\delta_1} \geq u_{\delta_2}$ in $\Omega^{\delta_2}$ whenever $0 < \delta_1 < \delta_2 \leq \gamma_0$, where $u_{\delta_i}$ is the unique positive solution of (2.15) with $\delta = \delta_i$, $i = 1, 2$. It follows that
\[ w(x) := \lim_{\delta \to 0^+} u_{\delta}(x) \]
is well-defined on $\Omega\setminus\{0\}$, and
\[ u_\delta(x) \leq w(x) \quad \forall x \in \Omega^\delta, \quad w(x) \leq U(x) \quad \forall x \in \Omega \setminus \{0\}. \]
The regularity of elliptic equations implies that $w \in C^2(\Omega \setminus \{0\})$ is a positive solution of (1.1) satisfying $w|_{\partial \Omega} = 0$. Suppose that $v$ is an arbitrary positive solution of (1.1). For any small $\delta > 0$, by Lemma 2.1, we have $u_\delta \leq v \in \Omega^\delta$. Letting $\delta \to 0$ we obtain $w \leq v$ in $\Omega \setminus \{0\}$. This shows that $w$ is the minimal positive solution of (1.1).

We are now ready to prove the first inequality in (2.6). Denote $B_{R, \gamma}(0) = \{x \in \mathbb{R}^N : \gamma < |x| < R\}$, and fix $R_0 > 0$ such that $B_{R_0}(0) \subset \Omega$. For a given $\lambda > (\frac{N-2}{2})^2$, using Lemma 2.3 with $\Omega$ replaced by $B_{R_0}(0)$ we see that there exists a small positive constant $\gamma_0$ such that
\[ \lambda > \lambda_1 \left[ \frac{1}{|x|^2}, B_{R_0, \gamma_0}(0) \right], \quad \frac{3\gamma_0}{2} < R_0. \]
By the monotone property of the first eigenvalue with respect to the domain, $\lambda > \lambda_1 \left[ \frac{1}{|x|^2}, B_{R, \gamma_0}(0) \right]$ when $R \geq R_0$. So, the associated problem
\[
\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & x \in B_{R, \gamma_0}(0), \\
u = 0, & x \in \partial B_{R, \gamma_0}(0)
\end{cases}
\]
has a unique positive solution, and we denote it by $U_{R, \gamma_0}(x)$. It is clear that $U_{R, \gamma_0}(x)$ is radially symmetric. For convenience, we also denote $U_{R, \gamma_0}(x)$ by $U_{R, \gamma_0}(r)$ with $r = |x|$. By Lemma 2.1, for arbitrary $R_1 > R_2 \geq R_0$ we have
\[ U_{R_1, \gamma_0}(x) \geq U_{R_2, \gamma_0}(x) \quad \forall x \in B_{R_2, \gamma_0}(0). \]
Let $\Theta_{\delta}$ be given by
\[ \Theta_{\delta}(x) = \left( \frac{\gamma_0}{\delta} \right)^{\frac{2+p}{2-p}} U_{\frac{2-p}{2} R_0, \gamma_0} \left( \frac{\gamma_0}{\delta} x \right), \quad x \in B_{R_0, \delta}(0), \quad \delta \leq \gamma_0. \]
For any $0 < \delta \leq \gamma_0$, a straightforward calculation gives
\[
\begin{cases}
-\Delta \Theta_{\delta} = \lambda \frac{\Theta_{\delta}}{|x|^2} - |x|^\sigma \Theta_{\delta}^p, & x \in B_{R_0, \delta}(0), \\
\Theta_{\delta} = 0, & |x| = \delta, \\
\Theta_{\delta} = 0, & |x| = R_0.
\end{cases}
\]
Since $B_{R_0}(0) \subset \Omega$, it follows from Lemma 2.1 that
\[ \Theta_{\delta}(x) \leq w(x) \quad \forall x \in B_{R_0, \delta}(0). \]
For any $0 \in (0, \gamma_0]$ and $|x| = \frac{3\delta}{2}$, we thus have
\[ w(x) \geq \Theta_{\delta}(x) = \left( \frac{\gamma_0}{\delta} \right)^{\frac{2+p}{2-p}} U_{\frac{2-p}{2} R_0, \gamma_0} \left( \frac{\gamma_0}{\delta} x \right) = \left( \frac{\gamma_0}{\delta} \right)^{\frac{2+p}{2-p}} U_{\frac{2-p}{2} R_0, \gamma_0} \left( \frac{3}{2} \frac{\gamma_0}{\delta} \right). \]
By (2.16), it is easy to see that \( U_{\frac{\gamma_0}{2} R_0, 3 \gamma_0} \geq U_{R_0, \gamma_0} \) for \( \delta \in (0, \gamma_0] \). Therefore, for \( \delta \in (0, \gamma_0] \) and \( |x| = \frac{3\delta}{2} \), we have
\[
w(x) \geq C \left( \frac{\gamma_0}{\delta} \right)^{\frac{2+\sigma}{p-1}}, \quad \text{with } C = U_{R_0, \gamma_0} \left( \frac{3}{2} \gamma_0 \right).
\]

This implies that there exist two constants \( C_2 > 0 \) and \( \delta_0 > 0 \) such that
\[
w(x) \geq C_2 |x|^{-\frac{2+\sigma}{p-1}}, \quad \forall x \in B_{\delta_0}(0) \backslash \{0\}.
\]

Since \( w \) is the minimal positive solution, it follows that
\[
u(x) \geq C_2 |x|^{-\frac{2+\sigma}{p-1}}, \quad \forall x \in B_{\delta_0}(0) \backslash \{0\},
\]
where \( u \) is an arbitrary positive solution of (1.1).

\[\square\]

2.4. The Dirichlet problem. In this subsection we consider the Dirichlet problem
\[
(2.17) \quad \begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^{\sigma} u^p, & x \in \Omega \backslash \{0\}, \\
u = 0, & x \in \partial \Omega.
\end{cases}
\]

**Proposition 2.5.** When \( \lambda > \left( \frac{N-2}{2} \right)^2 \), (2.17) has a unique positive solution.

**Proof.** In Section 2.3, we have proved that (2.17) has a minimal positive solution \( w \). If \( u(x) \) is any other positive solution of (2.17), then \( w(x) \leq u(x) \leq U(x) \), where \( U(x) \) is the maximal solution of (1.1). It now follows from (2.6) that there exist positive constants \( C_1, C_2 \) and \( \delta_0 \) such that
\[
C_2 |x|^{-\frac{2+\sigma}{p-1}} \leq w(x) \leq u(x) \leq U(x) \leq C_1 |x|^{-\frac{2+\sigma}{p-1}}, \quad 0 < |x| \leq \delta_0.
\]

Since \( u \) and \( w \) are positive in \( \Omega \backslash \{0\} \) and they vanish on \( \partial \Omega \), by virtue of Hopf’s boundary Lemma, there exists \( C_3 \geq 1 \) such that
\[
u(x) \leq C_3 w(x) \quad \text{for all } x \in \Omega \text{ close to } \partial \Omega.
\]

Therefore we can find a constant \( C \geq 1 \) such that
\[
w(x) \leq u(x) \leq C w(x) \quad \forall x \in \Omega \backslash \{0\}.
\]

Suppose \( w \neq u \). Then by the strong maximum principle it is necessary that
\[
w(x) < u(x) \quad \forall x \in \Omega \backslash \{0\}.
\]

We now define
\[
v = w - \frac{1}{2C} (u - w),
\]
and use a convex function trick of [11, 12] to deduce a contradiction.

We have
\[
w > v \geq \frac{C + 1}{2C} w, \quad \frac{2C}{2C + 1} v + \frac{1}{2C + 1} u = w.
\]

Let
\[
f(x, t) = -\lambda \frac{t}{|x|^2} + |x|^{\sigma} t^p, \quad |x| > 0.
\]

Then for \( t > 0 \) and \( x \neq 0 \),
\[
f_{u}(x, t) = p(p - 1)|x|^{\sigma} t^{p-2} > 0.
\]

So,
\[
f(x, w) \leq \frac{2C}{2C + 1} f(x, v) + \frac{1}{2C + 1} f(x, u),
\]
that is
\[
f(x, w) \leq \frac{1}{2C} f(x, u) - (1 + \frac{1}{2C}) f(x, u).
\]
Hence,
\[-\Delta v = -(1 + \frac{1}{2C})f(x, w) + \frac{1}{2C}f(x, u) \geq -f(x, v).\]

Since $v > 0$ in $\Omega \setminus \{0\}$ and $v = 0$ on $\partial \Omega$, we have, by Lemma 2.1,
\[(2.18) \quad u_\delta \leq v, \quad \forall x \in \Omega^\delta,\]
where $u_\delta$ is the unique positive solution of (2.15). Letting $\delta \to 0$ in (2.18) we obtain $w \leq v$, which is a contradiction to $w > v$. \hfill \Box

**Remark 2.6.** If $u = 0$ on $\partial \Omega$ is replaced by $u = c$ on $\partial \Omega$ in (2.17), where $c$ is a positive constant, the conclusion of Proposition 2.5 still holds; this can be proved by simple modifications of the above proof.

2.5. **On the condition** $\sigma > -2$. Lemma 2.4 indicates that when $\sigma > -2$ and $\lambda > (N - 2)^2/4$, any positive solution $u(x)$ of (1.1) must blow up as $|x| \to 0$. Here we show that this is no longer the case if $\sigma \leq -2$.

**Proposition 2.7.** If $\sigma \leq -2$ and $\lambda > (N - 2)^2/4$, then any positive solution $u(x)$ of (1.1) stays bounded near $x = 0$.

**Proof.** Suppose that $u$ is an arbitrary positive solution of (1.1). For any $x_0 \in \Omega \setminus \{0\}$ satisfying $d(x_0, \partial \Omega) > \frac{|x_0|}{2}$, denote
\[\mathbb{D}(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < \frac{|x_0|}{2}\}.\]

Since $\sigma \leq -2$, then
\[(2.19) \quad -\Delta u \leq 4\lambda|x_0|^{-2}u - 3^\sigma 2^{-\sigma}|x_0|^\sigma u^p, \quad \forall x \in \mathbb{D}(x_0).\]

Let
\[V(x) := |x_0|^\frac{2+\sigma}{p-1} u \left(x_0 + \frac{x_0}{2} x\right), \quad x \in B_1(0).\]

Inequality (2.19) implies (2.14). It is well known that
\[(2.20) \quad \begin{cases} -\Delta u = \lambda u - 2^{-2-\sigma} 3^\sigma u^p, & x \in B_1(0), \\ u = +\infty, & x \in \partial B_1(0) \end{cases}\]
has a unique positive solution, which we denote by $u_\infty$. By (2.20) and (2.14), we have
\[V(x) \leq u_\infty(x), \quad \forall x \in B_1(0).\]

Taking $x = 0$ we obtain
\[|x_0|^\frac{2+\sigma}{p-1} u(x_0) \leq u_\infty(0).\]

Note that $\sigma \leq -2$ implies $\frac{2+\sigma}{p-1} \leq 0$. By the arbitrariness of $x_0$, $u(x)$ is bounded near the origin. \hfill \Box

**Remark 2.8.** The restriction $\lambda > (N - 2)^2/4$ in Proposition 2.5 is unnecessary; a simple modification of the proof shows that the conclusion holds for any $\lambda \in \mathbb{R}^1$. Note also that the proof actually shows that $u(x) \leq C|x|^{-\frac{2+\sigma}{p-1}}$ for all $x$ near 0.
3. Proof of Theorem 1.1

Let us recall that (1.1) has a minimal positive solution \( w(x) \) and a maximal positive solution \( U(x) \), and

\[
\lim_{\delta \to 0} u_{\delta} = w, \quad \lim_{\delta \to 0} U_{\delta} = U,
\]

where \( u_{\delta} \) and \( U_{\delta} \) denote the unique positive solution of

\[
-\Delta u = \frac{\lambda}{|x|^2} u - |x|^\sigma u^p \quad \text{in } \Omega_{\delta}^\delta, \quad u|_{\partial\Omega^\delta} = 0
\]

and

\[
-\Delta u = \frac{\lambda}{|x|^2} u - |x|^\sigma u^p \quad \text{in } \Omega_{\delta}^\delta, \quad u|_{\partial\Omega^\delta} = +\infty,
\]

respectively.

If we take

\[
\xi = \left[ \lambda + 2 + \sigma \left( \frac{2 + \sigma}{p - 1} \right) + 2 - N \right]^{1/(p-1)} = \ell^{1/(p-1)},
\]

then a simple calculation shows that

\[
\psi(x) := \xi |x|^{-\frac{2+\sigma}{p-1}}
\]

satisfies (1.1) over \( \mathbb{R}^N \setminus \{0\} \). To stress the dependence of \( \xi \) on \( \lambda \), we may write \( \xi = \xi(\lambda) \). We note that when \( \lambda > (N - 2)^2/4 \),

\[
\xi(\lambda) > \xi((N - 2)^2/4) \geq 0.
\]

Let \( u(x) \) be an arbitrary positive solution of (1.1). Since

\[
w(x) \leq u(x) \leq U(x) \quad \text{in } \Omega \setminus \{0\} \quad \text{and } \xi = \ell^{1/(p-1)},
\]

Theorem 1.1 will follow if we can show

\[
\liminf_{|x| \to 0} |x|^{\frac{2+\sigma}{p-1}} w(x) \geq \xi, \quad \limsup_{|x| \to 0} |x|^{\frac{2+\sigma}{p-1}} U(x) \leq \xi.
\]

We prove these inequalities by two lemmas.

**Lemma 3.1.** For the minimal positive solution \( w(x) \), we have

\[
\liminf_{|x| \to 0} |x|^{\frac{2+\sigma}{p-1}} w(x) \geq \xi.
\]

**Proof.** Clearly (3.1) is equivalent to

\[
\limsup_{|x| \to 0} \frac{\psi(x)}{w(x)} \leq 1.
\]

Fix \( \delta > 0 \) small so that \( B_{\delta}(0) \subset \Omega \). By Proposition 2.5,

\[
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, \quad x \in B_{\delta}(0) \setminus \{0\},
\]

\[
u = 0, \quad x \in \partial B_{\delta}(0)
\]

has a unique positive solution, and we denote it by \( \zeta(x) \). Denote \( B_{\delta,\tau}(0) = \{x : \tau < |x| < \delta\} \). When \( \tau > 0 \) is sufficiently small, by Lemma 2.3,

\[
\lambda_1 \left[ \frac{1}{|x|^2} \right]_{B_{\delta,\tau}(0)} < \lambda.
\]

Let \( u_{\delta,\tau} \) denote the unique positive solution of

\[
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, \quad x \in B_{\delta,\tau}(0),
\]

\[
u = 0, \quad x \in \partial B_{\delta,\tau}(0).
\]
Then by the uniqueness of $\zeta(x)$,

$$\zeta(x) = \lim_{\tau \to 0} u_{\delta, \tau}(x).$$

By Lemma 2.1 we have

$$w(x) \geq u_{\delta, \tau}(x), \quad \forall x \in B_{\delta, \tau}(0).$$

It follows that

$$w(x) \geq \zeta(x), \quad \forall x \in B_\delta(0) \setminus \{0\}.$$ Therefore, (3.2) will follow if we can show

$$\lim_{|x| \to 0} \frac{\psi(x)}{\zeta(x)} = 1.$$ (3.5)

To prove (3.5), we denote

$$\beta := \inf_{x \in B_\delta(0) \setminus \{0\}} \frac{\psi(x)}{\zeta(x)}.$$ Note that $\psi(x)$ is a supersolution of (3.4). By Lemma 2.1, we have $\psi \geq u_{\delta, \tau}$ and hence $\psi(x) \geq \zeta(x)$, $\forall x \in B_\delta(0) \setminus \{0\}$, which implies $\beta \geq 1$.

We now complete the proof of (3.5) in two steps below.

**Step 1:** $\beta = 1$ implies (3.5).

We first note that $\psi$ and $\zeta$ are radially symmetric. By abusing notation for convenience, we will also write $\psi = \psi(r)$ and $\zeta = \zeta(r)$ with $r = |x|$. Since $\psi \geq \zeta$ in $B_\delta(0) \setminus \{0\}$ and $\psi > \zeta$ on $\partial B_\delta(0)$, by the strong maximum principle we deduce

$$\psi > \zeta$$ in $B_\delta(0) \setminus \{0\}.$ (3.6)

Hence $\beta = 1$ implies

$$\liminf_{r \to 0} \frac{\psi(r)}{\zeta(r)} = 1,$$

and (3.5) will follow if we can show $\limsup_{r \to 0} \frac{\psi(r)}{\zeta(r)} = 1$.

Arguing indirectly we assume that $\limsup_{r \to 0} \frac{\psi(r)}{\zeta(r)} > 1$. It follows that there exists $r_0 \in (0, \delta)$ such that $\frac{\psi(r)}{\zeta(r)}$ takes a local minimum at $r_0$ and $\frac{\psi(r_0)}{\zeta(r_0)} > 1$ due to (3.6). This implies in particular

$$\psi'(r_0) - \frac{\psi(r_0)}{\zeta(r_0)} \zeta'(r_0) = 0.$$ (3.7)

For any sufficiently small $\tau \in (0, r_0)$, let $\psi_\tau$ denote the unique positive solution of

$$\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & x \in B_{r_0, \tau}(0), \\
u = \psi(r_0), & |x| = r_0, \\
u = 0, & |x| = \tau.
\end{cases}$$ (3.8)

Since $\frac{\psi(r_0)}{\zeta(r_0)} > 1$, it is easily seen that $\frac{\psi(r_0)}{\zeta(r_0)} \zeta(r)$ is a supersolution of (3.8). Therefore, by Lemma 2.1, we have

$$\frac{\psi(r_0)}{\zeta(r_0)} \zeta(r) \geq \psi_\tau(r), \quad r \in (\tau, r_0).$$

By Remark 2.6, $\psi$ is the unique positive solution of

$$\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & 0 < |x| < r_0, \\
u = \psi(r_0), & |x| = r_0
\end{cases}$$ (3.9)
and hence 
\[ \psi = \lim_{\tau \to 0} \psi_\tau. \]

Therefore, we have
\[ \frac{\psi(r_0)}{\zeta(r_0)} \zeta(r) \geq \psi(r), \quad r \in (0, r_0). \]

Since \( \frac{\psi(r_0)}{\zeta(r_0)} \zeta(r) \) is a strict supersolution of (3.9), by the strong maximum principle,
\[ \frac{\psi(r_0)}{\zeta(r_0)} \zeta(r) > \psi(r), \quad r \in (0, r_0). \]

We may now apply Hopf’s boundary lemma to obtain
\[ \frac{\psi(r_0)}{\zeta(r_0)} \zeta'(r_0) - \psi'(r_0) < 0, \]
which is in contradiction to (3.7). This completes the proof of Step 1.

**Step 2:** \( \beta > 1 \) leads to a contradiction.

By the definition of \( \beta \), we have
\[ (3.10) \quad \psi(r) \geq \beta \zeta(r), \quad \forall r \in (0, \delta). \]

We claim that
\[ (3.11) \quad \psi(r) > \beta \zeta(r), \quad \forall r \in (0, \delta). \]

Otherwise there exists \( r_0 \in (0, \delta) \) such that
\[ \psi(r_0) = \beta \zeta(r_0). \]

Since \( \beta > 1 \), \( \beta \zeta(x) \) is a supersolution of (3.8), which implies
\[ \beta \zeta(r) \geq \psi_\tau(r), \quad \forall r \in (\tau, r_0). \]

Since \( \psi = \lim_{\tau \to 0} \psi_\tau \), it follows that
\[ \beta \zeta(r) \geq \psi(r), \quad \forall r \in (0, r_0). \]

As \( \beta \zeta(x) \) is a strict supersolution of (3.9), by the strong maximum principle,
\[ \beta \zeta(r) > \psi(r), \quad \forall r \in (0, r_0), \]
which is in contradiction to (3.10). This proves (3.11), which implies that
\[ \beta = \liminf_{r \to 0} \frac{\psi(r)}{\zeta(r)}. \]

We next show that
\[ (3.12) \quad \beta = \lim_{r \to 0} \frac{\psi(r)}{\zeta(r)}. \]

Otherwise we must have \( \beta < \limsup_{r \to 0} \frac{\psi(r)}{\zeta(r)} \). As before, this implies the existence of \( r_0 \in (0, \delta) \) such that \( \frac{\psi(r)}{\zeta(r)} \) takes a local minimum at \( r_0 \) and hence
\[ \psi'(r_0) - \frac{\psi(r_0)}{\zeta(r_0)} \zeta'(r_0) = 0. \]

Recall that we also have \( \frac{\psi(r_0)}{\zeta(r_0)} > \beta > 1 \). Therefore we can repeat the argument in Step 1 to derive a contradiction. This proves (3.12).
We now make use of (3.12) to derive a contradiction. Set $\alpha = \frac{\beta + 1}{2}$ and let $c \in (0, 1)$ be a constant with its value to be further specified later. Define

$$v = \alpha \zeta(x) - \frac{1}{c} [\psi(x) - \alpha \zeta(x)] = \frac{c + 1}{c} \alpha \zeta(x) - \frac{1}{c} \psi(x),$$

and

$$f(x, t) = -\lambda \frac{t}{|x|^\sigma} + |x|^\sigma t^p.$$

Then, $f_t(x, t) \geq 0$ for $x \neq 0$ and $t \geq 0$. Since $\alpha > 1$, it is easily seen that

$$-\Delta (\alpha \zeta) \geq -f(x, \alpha \zeta).$$

So, we have

$$-\Delta v = \frac{c + 1}{c} (-\Delta \alpha \zeta) + \frac{1}{c} \Delta \psi \geq -f(x, v), \quad 0 < |x| < \delta.$$

Using (3.12), for any small $\eta \in (0, \beta)$ satisfying $\beta + 2\eta - 1 < \beta + 1$, we can find $\gamma \in (0, \delta)$ such that

$$\psi(x) < (\beta + \eta) \zeta(x), \quad 0 < |x| \leq \gamma.$$

We now further restrict $c$ to satisfy $c \in (\frac{\beta + 2\eta - 1}{\beta + 1}, 1)$. Then

$$(c + 1)\alpha = (c + 1) \frac{\beta + 1}{2} > (\beta + \eta),$$

and it follows that

$$v(x) = \alpha \zeta(x) - \frac{1}{c} (\psi - \alpha \zeta(x)) = \frac{1}{c} [(c + 1)\alpha \zeta(x) - \psi(x)] > 0, \quad 0 < |x| \leq \gamma.$$

From (3.13) and (3.14) we see that $v$ is a supersolution of

$$\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & 0 < |x| < \gamma, \\
u = 0, & |x| = \gamma.
\end{cases}$$

By Proposition 2.5, (3.15) has a unique positive solution, which we denote by $u(x; \gamma)$. Let $z(x)$ be given by

$$z(x) = \left( \frac{\delta}{\gamma} \right)^{\frac{2+\sigma}{p+1}} \zeta \left( \frac{\delta}{\gamma} x \right).$$

Then, by direct calculation, $z(x)$ also satisfies (3.15). By uniqueness,

$$u(x; \gamma) \equiv \left( \frac{\delta}{\gamma} \right)^{\frac{2+\sigma}{p+1}} \zeta \left( \frac{\delta}{\gamma} x \right).$$

Define

$$\psi(x; \gamma) = \left( \frac{\delta}{\gamma} \right)^{\frac{2+\sigma}{p+1}} \psi \left( \frac{\delta}{\gamma} x \right).$$

It follows from the definition of $\psi(x)$ that $\psi(x; \gamma) \equiv \psi(x)$. Due to (3.12), we thus have

$$\lim_{|x| \to 0} \frac{\psi(x)}{u(x; \gamma)} = \lim_{|x| \to 0} \frac{\psi(x; \gamma)}{u(x; \gamma)} = \lim_{|x| \to 0} \frac{\psi(x)}{\zeta(x)} = \beta.$$

Since $v$ is a supersolution of (3.15), as before it can be easily shown that

$$v(x) \geq u(x; \gamma) \quad \forall 0 < |x| < \gamma,$$
that is,
\[
\alpha \zeta(x) - \frac{1}{c}(\psi(x) - \alpha \zeta(x)) \geq u(x; \gamma), \quad \forall 0 < |x| < \gamma.
\]
It follows that
\[
(3.17) \quad \frac{\zeta(x)}{u(x; \gamma)} - \frac{1}{c} \left[ \frac{\psi(x)}{u(x; \gamma)} - \frac{\zeta(x)}{u(x; \gamma)} \right] \geq 1, \quad \forall 0 < |x| < \gamma.
\]
By (3.16), we have
\[
\lim_{|x| \to 0} \frac{\zeta(x)}{u(x; \gamma)} = \lim_{|x| \to 0} \frac{\zeta(x)}{\psi(x)} = 1.
\]
Letting $|x| \to 0$ in (3.17), we deduce
\[
\alpha - \frac{1}{c}(\beta - \alpha) \geq 1,
\]
which, in view of the definition of $\alpha$, is equivalent to $c \geq 1$. But this is a contradiction to $c < 1$. The proof of Step 2 and hence the proof of the lemma is now complete. \hfill \Box

**Lemma 3.2.** For the maximal positive solution $U(x)$ we have
\[
(3.18) \quad \limsup_{|x| \to 0} |x|^\frac{2}{p-1} U(x) \leq \xi.
\]
Proof. Clearly (3.18) is equivalent to
\[
(3.19) \quad \limsup_{|x| \to 0} \frac{U(x)}{\psi(x)} \leq 1.
\]
Choose $\delta > 0$ small so that $B_{\delta}(0) \subset \Omega$, and for all large positive integer $n$, let $\rho_n(x)$ be the unique positive solution to
\[
\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & \frac{1}{n} < |x| < \delta, \\
u = +\infty, & |x| = \delta, \\
u = +\infty, & |x| = \frac{1}{n}.
\end{cases}
\]
Then
\[
\rho(x) := \lim_{n \to \infty} \rho_n(x)
\]
is a positive solution of
\[
(3.20) \quad \begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & 0 < |x| < \delta, \\
u = +\infty, & |x| = \delta.
\end{cases}
\]
By Lemma 2.1,
\[
\rho_n \geq U \quad \text{and hence} \quad \rho(x) \geq U(x), \quad \forall x \in B_\delta(0) \setminus \{0\}.
\]
As in the proof of Lemma 3.1, we let $\zeta(x)$ be the unique positive solution of (3.3), and recall that $\psi(x) \geq \zeta(x)$ for $0 < |x| \leq \delta$. It is now clear that (3.19) will follow if we can show
\[
(3.21) \quad \lim_{|x| \to 0} \frac{\rho(x)}{\zeta(x)} = 1.
\]
Define
\[
\beta := \inf_{0 < |x| < \delta} \frac{\rho(x)}{\zeta(x)}.
\]
Since $\rho \geq U \geq \psi \geq \zeta$, we obviously have $\beta \geq 1$. We note that both $\rho(x)$ and $\zeta(x)$ are radially symmetric, so as before we will also write $\rho = \rho(r)$ and $\zeta = \zeta(r)$ with $r = |x|$.
The arguments below are parallel to those in the proof of Lemma 3.1, with some significant variations appear near the end.

**Step 1:** $\beta = 1$ implies (3.21).

By the strong maximum principle we have $\rho(r) > \zeta(r)$ for $r \in (0, \delta)$. Hence $\beta = 1$ implies $\lim\inf_{r \to 0} \rho(r)/\zeta(r) = 1$. To complete the proof it suffices to show $\lim\sup_{r \to 0} \rho(r)/\zeta(r) = 1$. If on the contrary $\lim\sup_{r \to 0} \rho(r)/\zeta(r) > 1$, then we can find $r_0 \in (0, \delta)$ such that $\frac{\rho(r)}{\zeta(r)}$ has a local minimum at $r = r_0$. Note that we also have $\frac{\rho(r)}{\zeta(r)} > 1$. Hence we can use the same argument in Step 1 of the proof of Lemma 3.1 to obtain a contradiction. This completes the proof of Step 1 here.

**Step 2:** $\beta > 1$ leads to a contradiction.

By the definition of $\beta$ we have $\rho(r) \geq \beta \zeta(r)$ for $r \in (0, \delta)$. The same reasoning as in Step 2 of the proof of Lemma 3.1 yields $\rho(r) > \beta \zeta(r)$ for $r \in (0, \delta)$. Hence necessarily

$$\beta = \lim\inf_{r \to 0} \rho(r)/\zeta(r).$$

We claim that

$$\beta = \lim_{r \to 0} \rho(r)/\zeta(r).$$

Otherwise we have $\beta < \lim\sup_{r \to 0} \rho(r)/\zeta(r)$, and hence there exists $r_0 \in (0, \delta)$ such that $r_0$ is a local minimum point of $\frac{\rho(r)}{\zeta(r)}$ and $\frac{\rho(r)}{\zeta(r)} > \beta > 1$. We can now obtain a contradiction as before. This proves (3.22).

Next we use (3.22) to derive a contradiction. Define $\alpha := \frac{\beta + 1}{2}$ and let $c \in (0, 1)$ be a constant to be further specified later. Let $z(x)$ be given by

$$z(x) := \alpha \zeta(x) - \frac{1}{c} [\rho(x) - \alpha \zeta(x)].$$

Then

$$-\Delta z \geq \lambda \frac{z}{|x|^2} - |x|^\sigma z^p.$$

Due to (3.22), for $\eta \in (0, \beta)$ small, and $c \in (\frac{\beta + 2\eta - 1}{\beta + 1}, 1)$, there exists $\gamma \in (0, \delta)$ such that

$$z(x) = \alpha \zeta(x) - \frac{1}{c} [\rho(x) - \alpha \zeta(x)] > 0, \quad 0 < |x| \leq \gamma.$$

It is easily seen, as before, that

$$\zeta(x; \gamma) := \left( \frac{\delta}{\gamma} \right)^{\frac{2 + \sigma}{\delta}} \zeta \left( \frac{\delta}{\gamma} x \right)$$

is the unique positive solution of

$$\begin{cases}
-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, & 0 < |x| < \gamma, \\
u = 0, & |x| = \gamma.
\end{cases}$$

By (3.23) and (3.24), $z$ is a positive supersolution of (3.25). By Lemma 2.1 we can easily show that

$$z(x) \geq \zeta(x; \gamma), \quad \forall x \in B_\gamma(0) \setminus \{0\}.$$
It is easily seen that $\rho_n(x; \gamma)$ solves
\[
\begin{aligned}
-\Delta u &= \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, \\
|u| &= +\infty, \\
|u| &= \frac{\gamma}{n\delta}, \\
|u| &\in \left\{ \frac{\gamma}{n\delta}, \gamma \right\}.
\end{aligned}
\]

By Lemma 2.1,
\[
\rho(x) \leq \rho_n(x; \gamma) \quad \text{for} \quad \frac{\gamma}{n\delta} < |x| < \gamma.
\]

Letting $n \to \infty$, we obtain
\[
\rho(x) \leq \rho(x; \gamma) := \lim_{n \to \infty} \rho_n(x; \gamma) = \left( \frac{\delta}{\gamma} \right)^{\frac{2-n}{p-1}} \rho \left( \frac{\delta}{\gamma} x \right) \quad \forall x \in B_\gamma(0) \setminus \{0\}.
\]

Together with $\lim_{|x| \to 0} \frac{\rho(x)}{\xi(x)} = \beta$, we obtain
\[
(3.27) \quad \limsup_{|x| \to 0} \frac{\xi(x)}{\rho(x; \gamma)} \leq \frac{1}{\beta}.
\]

It is clear that (3.22) also implies
\[
(3.28) \quad \lim_{|x| \to 0} \frac{\rho(x; \gamma)}{\xi(x; \gamma)} = \beta.
\]

By (3.27) and (3.28), we infer that
\[
(3.29) \quad \limsup_{|x| \to 0} \frac{\xi(x)}{\xi(x; \gamma)} = \limsup_{|x| \to 0} \left[ \frac{\xi(x)}{\rho(x; \gamma)} \frac{\rho(x; \gamma)}{\xi(x; \gamma)} \right] \leq 1.
\]

By Lemma 2.1, it is easy to prove
\[
\xi(x) \geq \xi(x; \gamma), \quad \forall x \in B_\gamma(0) \setminus \{0\}.
\]

Thus
\[
(3.30) \quad \liminf_{|x| \to 0} \frac{\xi(x)}{\xi(x; \gamma)} \geq 1.
\]

From (3.29) and (3.30) we see
\[
(3.31) \quad \lim_{|x| \to 0} \frac{\xi(x)}{\xi(x; \gamma)} = 1.
\]

Therefore
\[
(3.32) \quad \lim_{|x| \to 0} \frac{\rho(x)}{\xi(x; \gamma)} = \lim_{|x| \to 0} \left[ \frac{\rho(x)}{\xi(x)} \frac{\xi(x)}{\xi(x; \gamma)} \right] = \beta.
\]

Combining (3.26), (3.31) and (3.32), we obtain
\[
\alpha - \frac{1}{c} (\beta - \alpha) \geq 1,
\]

which is equivalent to $c \geq 1$, a contradiction to $c \in \left( \frac{\beta + 2n - 1}{\beta + 1}, 1 \right)$. The proof is now complete. \(\square\)
REFERENCES


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EXACT SINGULAR BEHAVIOR OF POSITIVE SOLUTIONS